Ageing and dynamical symmetries in the kinetics of interface growth

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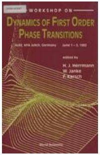
MH, **J.D. Noh** and **M. Pleimling**, Phys. Rev. **E85**, 030102(R) (2012)
MH & **S. Stoimenov**, Nucl. Phys. **B847**, 612 (2011);
J. Phys. **A46**, 245004 (2013); Symmetry **7**, 1595 (2015)
MH, Nucl. Phys. **B869**, 282 (2013); MH & **S. Rouhani**, J. Phys. **A46**, 494004 (2013)
MH & **X. Durang**, J. Stat. Mech. P05022 (2015) *& work in progress*

Overview :

1. Physical ageing & interface growth
2. Interface growth & universality classes
3. Linear responses and extensions of dynamical scaling
4. Shape of the scaling functions & LSI : building blocks
5. Conclusions

a word on history : first ideas on LSI $\hat{=}$ '*Schrödinger-invariance*'



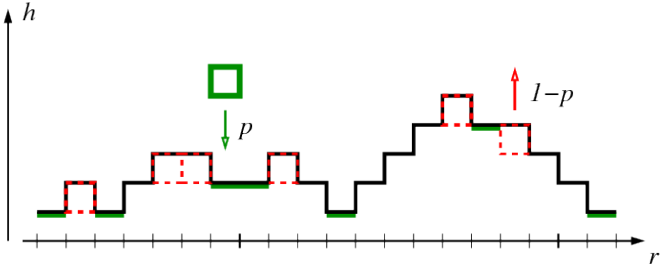
presented in **June 1992**, at Jülich Workshop
"Dynamics of first-order phase transitions"
proceedings edited by H.J. Herrmann, W. **Janke**, F. Karsch

relationship with ageing in simple magnets had to wait until 2001

1. Physical ageing & interface growth

Interface growth : deposition (evaporation) of particles on a substrate

$\rightarrow$ height profile $h(t, \mathbf{r})$ slope profile $\mathbf{u}(t, \mathbf{r}) = \nabla h(t, \mathbf{r})$



p = deposition prob.
$1 - p$ = evap. prob.

Questions :

* average properties of profiles & their fluctuations ?
* what about their relaxational properties ?
* are these also examples of physical ageing ?

? does dynamical scaling **always** exist ? are there extensions ?

Ageing magnets & growing interfaces : analogies

known & practically used since prehistoric times (metals, glasses)

systematic studies in physics since the 1970s $\Longrightarrow$ { **reproducible** / **universal** } Struik 78

Three defining properties of **ageing** :

1. slow relaxation (non-exponential !)
2. **no** time-translation-invariance (TTI)
3. dynamical scaling

without fine-tuning of parameters

* **collective** behaviour — **large** number of interacting degrees of freedom
* **algebraic** long-time behaviour

  length scale $L(t) \sim t^{1/z}$ dynamical exponent z
* described in terms of **universal** critical **exponents**
* very **few** relevant scaling operators
* justifies use of **simplified** **mathematical models**
* yet of **experimental significance**

Magnets

thermodynamic equilibrium state

order parameter $\phi(t,\mathbf{r})$

phase transition, at critical temperature T_c

variance :

$$\left\langle \left(\phi(t,\mathbf{r}) - \langle\phi(t)\rangle\right)^2 \right\rangle \sim t^{-2\beta/(\nu z)}$$

relaxation, after quench to $T \leq T_c$

autocorrelator

$$C(t,s) = \langle \phi(t,\mathbf{r})\phi(s,\mathbf{r})\rangle_c$$

Interfaces

growth continues forever

height profile $h(t,\mathbf{r})$

same generic behaviour throughout

roughness :

$$w(t)^2 = \left\langle \left(h(t,\mathbf{r}) - \overline{h}(t)\right)^2 \right\rangle \sim t^{2\beta}$$

relaxation, from initial substrate :

autocorrelator $C(t,s) =$

$$\left\langle \left(h(t,\mathbf{r}) - \overline{h}(t)\right)\left(h(s,\mathbf{r}) - \overline{h}(s)\right)\right\rangle$$

ageing scaling behaviour :

when $t,s\to\infty$, and $y := t/s > 1$ fixed, expect, with $\left\{ \begin{array}{l} \text{waiting time } s \\ \text{observation time } t > s \end{array}\right.$

$$C(t,s) = s^{-b} f_C(t/s) \quad \text{and} \quad f_C(y) \overset{y\to\infty}{\sim} y^{-\lambda_C/z}$$

b, β, ν and dynamical exponent z : **universal** & related to stationary state

autocorrelation exponent λ_C : **universal** & independent of stationary exponents

Magnets

exponent value $b = \begin{cases} 0 & ; \; T < T_c \\ 2\beta/\nu z & ; \; T = T_c \end{cases}$

Interfaces

exponent value $b = -2\beta$

models :

(a) gaussian field

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,(\nabla\phi)^2$

(b) **Ising model**

$\mathcal{H}[\phi] = -\frac{1}{2}\int \mathrm{d}\mathbf{r}\,\left[(\nabla\phi)^2 + \tau\phi^2 + \frac{g}{2}\phi^4\right]$

such that $\tau = 0 \leftrightarrow T = T_c$

dynamical Langevin equation (Ising) :

$$\partial_t\phi = -D\frac{\delta\mathcal{H}[\phi]}{\delta\phi} + \eta$$
$$= D\nabla^2\phi + \tau\phi + g\phi^3 + \eta$$

$\eta(t,\mathbf{r})$ is the usual white noise, $\langle\eta(t,\mathbf{r})\eta(t',\mathbf{r}')\rangle = 2T\delta(t-t')\delta(\mathbf{r}-\mathbf{r}')$

phase transition exactly solved $d = 2$

relaxation exactly solved $d = 1$

Onsager '44, Glauber '63, ...

(a) Edwards-Wilkinson (EW) :

$\partial_t h = \nu\nabla^2 h + \eta$

(b) **Kardar-Parisi-Zhang** (KPZ) :

$$\partial_t h = \nu\nabla^2 h + \frac{\mu}{2}(\nabla h)^2 + \eta$$

growth exactly solved $d = 1$

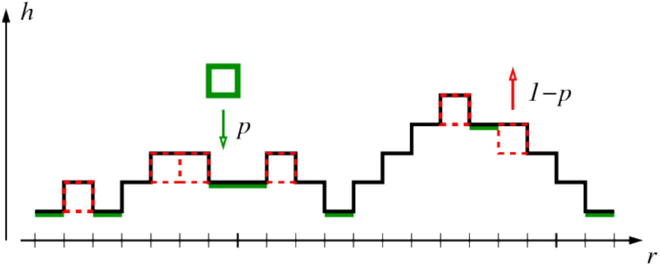
Sasamoto & Spohn '10

Calabrese & Le Doussal '11, ...

2. Interface growth & universality classes

deposition (evaporation) of particles on a substrate $\rightarrow$ height profile $h(t, \mathbf{r})$

generic situation : RSOS (**r**estricted **s**olid-**o**n-**s**olid) model — Kim & Kosterlitz 89



p = deposition prob.

$1 - p$ = evap. prob.

here $p = 0.98$

some universality classes :

(a) KPZ $\partial_t h = \nu \nabla^2 h + \frac{\mu}{2} (\nabla h)^2 + \eta$ — Kardar, Parisi, Zhang 86

(b) EW $\partial_t h = \nu \nabla^2 h + \eta$ — Edwards, Wilkinson 82

(c) Arcetri I $\partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) u + \nabla \eta,$ $\left\langle \int u^2 \right\rangle = 1$ — MH & Durang 15

(d) Arcetri II $\partial_t u = \nu \nabla^2 u + \mathfrak{z}(t) \nabla u + \nabla \eta,$ $\left\langle \int u^2 \right\rangle = 1$ — Durang & MH 16

(e) Arcetri III $\partial_t h = \nu \nabla^2 h + \mathfrak{z}(t) \nabla h + \eta,$ $\left\langle \int (\nabla h)^2 \right\rangle = 1$ — Durang & MH 16

η is a gaussian white noise with $\langle \eta(t, \mathbf{r}) \eta(t', \mathbf{r}') \rangle = 2\nu T \delta(t - t') \delta(\mathbf{r} - \mathbf{r}')$

Family-Viscek scaling on a spatial lattice of extent L^d : $\overline{h}(t) = L^{-d} \sum_j h_j(t)$

Family & Viscek 85

$$w^2(t;L) = \frac{1}{L^d} \sum_{j=1}^{L^d} \left\langle \left(h_j(t) - \overline{h}(t)\right)^2 \right\rangle = L^{2\alpha} f\left(tL^{-z}\right) \sim \begin{cases} L^{2\alpha} & ;\ \text{if } tL^{-z} \gg 1 \\ t^{2\beta} & ;\ \text{if } tL^{-z} \ll 1 \end{cases}$$

β : growth exponent, α : roughness exponent, $\boxed{\alpha = \beta z}$

Galilei-invariance $\Rightarrow$ $\boxed{\alpha + z = 2}$

two-time correlator :

limit $L \to \infty$

$$C(t,s;\mathbf{r}) = \left\langle \left(h(t,\mathbf{r}) - \langle \overline{h}(t)\rangle\right)\left(h(s,\mathbf{0}) - \langle \overline{h}(s)\rangle\right)\right\rangle = s^{-b} F_C\left(\frac{t}{s}, \frac{\mathbf{r}}{s^{1/z}}\right)$$

with ageing exponent : $\boxed{b = -2\beta}$

Kallabis & Krug 96

expect for $y = t/s \gg 1$: $F_C(y,\mathbf{0}) \sim y^{-\lambda_C/z}$ autocorrelation exponent

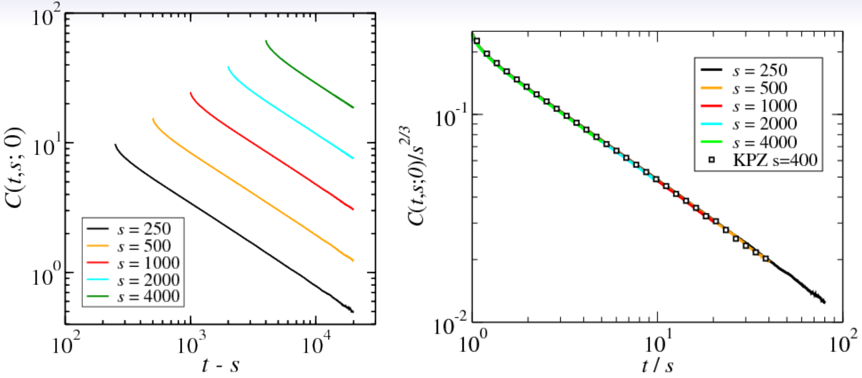
rigorous bound : $\boxed{\lambda_C \geq (d + zb)/2}$

Yeung, Rao, Desai 96 ; mh & Durang 15

KPZ class, to all orders in perturbation theory $\boxed{\lambda_C = d \text{ , if } d < 2}$

Krech 97

$1D$ relaxation dynamics, starting from an initially flat interface



observe all **3** properties of **ageing** : slow dynamics; no TTI; dynamical scaling

confirm **simple ageing** for the $1D$ KPZ universality class

confirm expected exponents $b = -2/3$, $\lambda_C/z = 2/3$ *pars pro toto*

Kallabis & Krug 96 ; Krech 97 ; Bustingorry *et al.* 07-10 ; Chou & Pleimling 10 ;
D'Aquila & Täuber 11/12 ; mh, Noh, Pleimling 12 . . .

Experiment : **universality** of interface exponents, KPZ class

model/system	d	z	β	α
KPZ	1	3/2	1/3	1/2
Ag electrodeposition	1		$\approx 1/3$	$\approx 1/2$
slow paper cumbustion	1	1.44(12)	0.32(4)	0.49(4)
liquid crystal (flat)	1	1.34(14)	0.32(2)	0.43(6)
liquid crystal (circular)	1	1.44(10)	0.334(3)	0.48(5)
cell colony growth	1	1.56(10)	0.32(4)	0.50(5)
(almost) isotrope colloïds	1		0.37(4)	0.51(5)
autocatalytic reaction front	1	1.45(11)	0.34(4)	0.50(4)
KPZ	2	1.63(3)	0.2415(15)	0.393(4)
	2	1.63(2)	0.241(1)	0.393(3)
CdTe/Si(100) film	2	1.61(5)	0.24(4)	0.39(8)
EW sedimentation	2		0(log)	0(log)
/electrodispersion	2			

experimental results from **several groups**, since 1999 (**mainly** since 2010)

3. Linear responses and extensions of dynamical scaling

extend Family-Viscek scaling to two-time responses :

analogue : TRM integrated response in magnetic systems

two-time integrated response : mh, Noh, Pleimling 12

* sample **A** with deposition rates $p_i = p \pm \epsilon_i$, up to time s,
* sample **B** with $p_i = p$ up to time s ;

then switch to common dynamics $p_i = p$ for all times $t > s$

$$\chi(t,s;\mathbf{r}) = \int_0^s \mathrm{d}u\, R(t,u;\mathbf{r}) = \frac{1}{L}\sum_{j=1}^{L}\left\langle \frac{h_{j+r}^{(\mathbf{A})}(t;s) - h_{j+r}^{(\mathbf{B})}(t)}{\epsilon_j}\right\rangle = s^{-a} F_\chi\left(\frac{t}{s}, \frac{|\mathbf{r}|^z}{s}\right)$$

with a : ageing exponent

expect for $y = t/s \gg 1$: $F_R(y,\mathbf{0}) \sim y^{-\lambda_R/z}$ autoresponse exponent

? Values of these exponents **?**

Effective action of the KPZ equation :

$$\mathcal{J}[\phi,\widetilde{\phi}] = \int \mathrm{d}t\mathrm{d}\mathbf{r}\ \left[\widetilde{\phi}\left(\partial_t\phi - \nu\nabla^2\phi - \frac{\mu}{2}(\nabla\phi)^2\right) - \nu T\widetilde{\phi}^2\right]$$

$\Longrightarrow$ Very special properties of KPZ in $d=1$ spatial dimension !

Exact critical exponents $\beta = 1/3,\ \zeta = 1/2,\ z = 3/2,\ \lambda_C = 1$ KPZ 86 ; KRECH 97

Special KPZ symmetry in 1D : let $v = \frac{\partial\phi}{\partial r}$, $\widetilde{\phi} = \frac{\partial}{\partial r}\left(\widetilde{p} + \frac{v}{2T}\right)$

$$\mathcal{J} = \int \mathrm{d}t\mathrm{d}r\ \left[\widetilde{p}\partial_t v - \frac{\nu}{4T}(\partial_r v)^2 - \frac{\mu}{2}v^2\partial_r\widetilde{p} + \nu T(\partial_r\widetilde{p})^2\right]$$

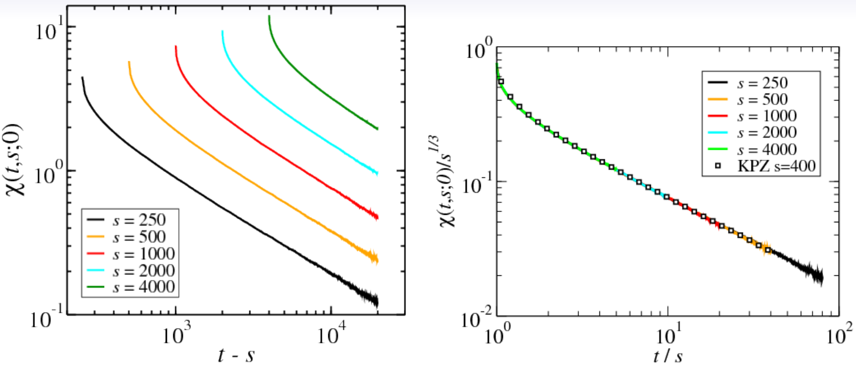
is invariant under **time-reversal** LVOV, LEBEDEV, PATON, PROCACCIA 93; FREY, TÄUBER, HWA 96

$$t \mapsto -t\ ,\ \ v(t,r) \mapsto -v(-t,r)\ ,\ \ \widetilde{p} \mapsto +\widetilde{p}(-t,r)$$

$\Rightarrow$ **fluctuation-dissipation relation** for $t \gg s$ $TR(t,s;r) = -\partial_r^2 C(t,s;r)$

find ageing exponents : $\lambda_R = \lambda_C = 1$ $1 + a = b + \frac{2}{z}$

relaxation of the integrated response,1D

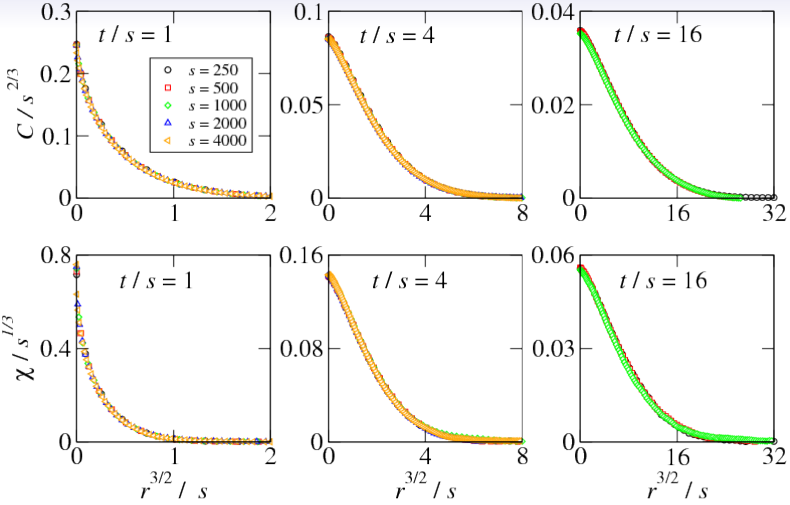


observe all **3** properties of **ageing** : $\begin{cases} \text{slow dynamics} \\ \text{no TTI} \\ \text{dynamical scaling} \end{cases}$

exponents $a = -1/3$, $\lambda_R/z = 2/3$, as expected from FDR

N.B. : numerical tests for 2 models in KPZ class

Simple ageing is also seen in time-space observables



correlator $C(t,s;r) = s^{2/3} F_C\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

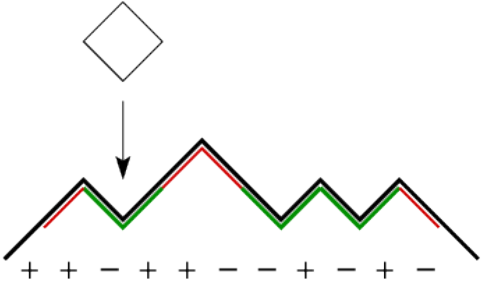
integrated response $\chi(t,s;r) = s^{1/3} F_\chi\left(\frac{t}{s}, \frac{r^{3/2}}{s}\right)$

} confirm $z = 3/2$

New dynamical universality classes : the **Arcetri** models

consider **RSOS**-adsorption process :

rigorous : continuum limit gives KPZ — Bertini & Giacomin 97



use **not** the heights $h_n(t) \in \mathbb{N}$ on a discrete lattice,
but rather the **slopes** $u_n(t) = \frac{1}{2}\left(h_{n+1}(t) - h_{n-1}(t)\right) = \pm 1$ — **RSOS**

? let $u_n(t) \in \mathbb{R}$, & impose a spherical constraint $\boxed{\sum_n \langle u_n(t)^2 \rangle \stackrel{!}{=} \mathcal{N}}$?

? consequences of the 'hardening' of a soft EW-interface by a 'spherical constraint' on the u_n ?

KPZ equation for height $h(t,r)$: $\partial_t h = \nu \partial_r^2 h + \frac{\mu}{2} (\partial_x h)^2 + \eta$

Burger's equation for slope $u(t,r) = \partial_r h(t,r)$:

$$\partial_t u = \nu \partial_r^2 u + \mu u \partial_r u + \partial_r \eta$$

model $\mathrm{A}I$: $\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1$

$\mathfrak{z}(t) \sim \langle\!\langle \partial_r u \rangle\!\rangle \sim \text{curvature}$

model $\mathrm{A}II$: $\partial_t u = \nu \partial_r^2 u + \mathfrak{z}(t) \partial_r u + \partial_r \eta, \qquad \int \mathrm{d}r \, \langle u^2 \rangle \sim 1$

$\mathfrak{z}(t) \sim \langle\!\langle u \rangle\!\rangle \sim \text{slope}$

model $\mathrm{A}III$: $\partial_t h = \nu \partial_r^2 h + \mathfrak{z}(t) \partial_r h + \eta, \qquad \int \mathrm{d}r \, \langle (\partial_r h)^2 \rangle \sim 1$

$\mathfrak{z}(t) \sim \langle\!\langle \partial_r h \rangle\!\rangle \sim \text{slope}$

? interface rough or smooth ?

? long-time properties and ageing behaviour ?

? does dynamical scaling resp. simple ageing always hold ?

model Arcetri I : exact solution :

$\omega(\mathbf{q}) = \sum_{a=1}^{d}(1 - \cos q_a)$, $\mathbf{q} \neq \mathbf{0}$

$$\widehat{h}(t,\mathbf{q}) = \widehat{h}(0,\mathbf{q})e^{-2t\omega(\mathbf{q})}\sqrt{\frac{1}{g(t)}} + \int_0^t \mathrm{d}\tau\, \widehat{\eta}(\tau,\mathbf{q})\sqrt{\frac{g(\tau)}{g(t)}}\, e^{-2(t-\tau)\omega(\mathbf{q})}$$

in terms of the auxiliary function $g(t) = \exp\left(-2\int_0^t \mathrm{d}\tau\, \mathfrak{z}(\tau)\right)$,
which satisfies Volterra equation

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, g(\tau) f(t-\tau) \quad , \quad f(t) := d\,\frac{e^{-4t} I_1(4t)}{4t}\left(e^{-4t} I_0(4t)\right)^{d-1}$$

* for $d=1$, identical to 'spherical spin glass', with $T = 2T_{\rm SG}$:
  hamiltonian $\mathcal{H} = -\frac{1}{2}\sum_{i,j} J_{ij} S_i S_j$; J_{ij} random matrix, its eigenvalues distributed according to Wigner's semi-circle law Cugliandolo & Dean 95
* also related to distribution of first gap of random matrices

Fyodorv, Perret & Schehr 15/16

Theory & Simulations : exponents in KPZ, EW and Arcetri I (AI) classes

model	d	z	β	a	b	λ_C	λ_R	
KPZ	1	$3/2$	$1/3$	$-1/3$	$-2/3$	1	1	
	2	1.61(2)	0.2415(15)	0.30(1)	$-0.483(3)$	1.97(3)	2.04(3)	[1]
	2	1.61(2)	0.241(1)		-0.483	1.91(6)		[2]
	2	1.61(5)	0.244(2)					[3]
	2	1.627(4)	0.229(6)					[4]
EW	< 2	2	$(2-d)/4$	$d/2-1$	$d/2-1$	d	d	
	2	2	0(log)	0	0	2	2	[5]
	> 2	2	0	$d/2-1$	$d/2-1$	d	d	
AI	< 2	2	$(2-d)/4$	$d/2-1$	$d/2-1$	$3d/2-1$	$3d/2-1$	
$T = T_c$	2	2	0(log)	0	0	2	2	[6]
	> 2	2	0	$d/2-1$	$d/2-1$	d	d	
$T < T_c$	d	2	$1/2$	$d/2-1$	-1	$d/2-1$	$d/2-1$	

Monte Carlo : [1] ODOR *et al.* 12, [2] HALPIN-HEALY *et al.* 14, [3] RODRIGUES *et al.* 15 ;

NPRG : [4] CANET *et al.* 11,12

exact : [5] PLEIMLING *et al.* 06, [6] MH & DURANG 15

Experiment $d = 1$: $b \approx -2/3$, $\lambda_C \approx 1$ TAKEUCHI & SANO 12

4. Shape of the scaling functions & LSI

? model-independent results on **shape** of universal scaling functions ?

? '*Natural*' starting point : conformal invariance at equilibrium ?

$\Longrightarrow$ considers space-dependent, angle-preserving rescalings

$\Longrightarrow$ 'normally' works for sufficiently 'local' interactions

? Ingredients for **time**-dependent critical phenomena ? CARDY 85, MH 93

* which sets of **time-space transformations** ?
* how should **scaling operators transform** ?
* include **dynamical scaling** and which kind of **Galilei-covariance** ?
* local or non-local transformations ?
* should LSI-transformations close into a Lie algebra ?
* **causality** properties ? covariant **response** functions, or **correlators** ?

here : outline of some of the main ingredients

(A) Standard (projective) conformal invariance at equilibrium

label coordinates as '*time*' t and '*space*' **r**

in $(1+1)D$ use complex variables $w = t + \mathrm{i}r$ and $\bar{w} = t - \mathrm{i}r$

Extend global dynamical scaling to local, projective transformations

$$w \mapsto \frac{\alpha w + \beta}{\gamma w + \delta} \;\; , \;\; \bar{w} \mapsto \frac{\bar{\alpha}\bar{w} + \bar{\beta}}{\bar{\gamma}\bar{w} + \bar{\delta}} \;\; , \;\; \alpha\delta - \beta\gamma = 1 \; , \; \bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\gamma} = 1$$

note : (i) translation-invariance in t, r **&** (ii) time-space rotation-invariance

Transformation of scaling operators $w \mapsto w'$ with $\dot{f}(w') \geq 0$ and

$$w = f(w') \;\; , \;\; \phi(w,\bar{w}) = \left(\frac{\mathrm{d}f(w')}{\mathrm{d}w'}\right)^{-\Delta} \left(\frac{\mathrm{d}\bar{f}(\bar{w}')}{\mathrm{d}\bar{w}'}\right)^{-\bar{\Delta}} \phi'(w',\bar{w}')$$

with $x = \Delta + \bar{\Delta}$ scaling dimension, $s = \Delta - \bar{\Delta}$ spin ('usually' $s = 0$)

at equilibrium, scalar ϕ has **a single** scaling dimension x.

infinitesimal generators $\ell_n = -w^{n+1}\partial_w - \Delta(n+1)w^n$

generators $X_n = \ell_n + \bar{\ell}_n$ and $Y_n = \ell_n - \bar{\ell}_n$ span conformal Lie algebra $\mathfrak{conf}(2)$

$$[X_n, X_m] = (n-m)X_{n+m}\ ,\ [X_n, Y_m] = (n-m)Y_{n+m}\ ,\ [Y_n, Y_m] = (n-m)X_{n+m} \qquad \text{(C)}$$

Invariant Schrödinger operator (Laplacian) $\mathcal{S} = 4\partial_w \partial_{\bar{w}}$

$$\begin{aligned} [\mathcal{S}, X_{-1}] \ = \ [\mathcal{S}, Y_{-1}] &= [\mathcal{S}, Y_0] \ = \ 0 \\ [\mathcal{S}, X_0] \ = \ -\mathcal{S}\ ,\ [\mathcal{S}, X_1] &= -2(w+\bar{w})\mathcal{S} - 8(\Delta\partial_{\bar{w}} + \bar{\Delta}\partial_w) \\ [\mathcal{S}, Y_1] &= -2(w-\bar{w})\mathcal{S} - 8(\Delta\partial_{\bar{w}} - \bar{\Delta}\partial_w) \end{aligned}$$

Lemma : If $\mathcal{S}\phi = 0$ and $\Delta = \bar{\Delta} = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. CARTAN 1909

$\boxed{\mathfrak{conf}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Co-variant two-point function (correlator) POLYAKOV 70

$$\langle \phi_1(t,r)\phi_2(0,0)\rangle = \delta_{x_1,x_2}\left(t^2+r^2\right)^{-x_1} = t^{-2x_1} f(r/t)\ ,\ f(u) \sim (1+u^2)^{-x_1} \qquad \text{(P)}$$

(B) Non-equilibrium dynamical scaling and ageing

consider Janssen-de Dominicis action

$$\mathcal{J}[\phi,\widetilde{\phi}] = \underbrace{\mathcal{J}_0[\phi,\widetilde{\phi}]}_{\textit{deterministic}} + \underbrace{\mathcal{J}_b[\widetilde{\phi}]}_{\textit{noise}}$$

Theorem : *Consideration of the* **'deterministic part'** *of the Janssen-de Dominicis action $\mathcal{J}$ permits to reconstruct the full time-dependent responses and correlators, from the dynamical symmetries of the 'deterministic part' $\mathcal{J}_0$.* Picone & mh 04

essential tool : Bargman superselection rule of 'deterministic part'

Example :

full response $R(t,s) = R_0(t,s)$ noiseless reponse, only obtained from $\mathcal{J}_0$

$\Rightarrow$ importance of analysis of two-time **response functions**, most simple form

Time-dependent critical phenomena & ageing

Characterised by **dynamical exponent** z : $t \mapsto tb^{-z}$, $\mathbf{r} \mapsto \mathbf{r}b^{-1}$

? Can one extend to **local** dynamical scaling, with $z \neq 1$?

For $z = 2$, example of the **Schrödinger group** : JACOBI 1842, LIE 1881

$$t \mapsto \frac{\alpha t + \beta}{\gamma t + \delta} \;,\; \mathbf{r} \mapsto \frac{\mathcal{D}\mathbf{r} + \mathbf{v}t + \mathbf{a}}{\gamma t + \delta} \;;\; \alpha\delta - \beta\gamma = 1$$

$\Rightarrow$ study **ageing** phenomena as paradigmatic example

essential : (i) **absence** of TTI & (ii) **Galilei**-invariance

Transformation $t \mapsto t'$ with $f(0) = 0$ and $\dot{f}(t') \geq 0$ and

$$t = f(t') \;,\; \phi(t) = \left(\frac{\mathrm{d}f(t')}{\mathrm{d}t'}\right)^{-x/z} \left(\frac{\mathrm{d}\ln f(t')}{\mathrm{d}t'}\right)^{-2\xi/z} \phi'(t')$$

out of equilibrium, have **2 distinct** scaling dimensions, x and ξ.

mean-field for **magnets** : expect $\begin{cases} \xi = 0 \text{ in ordered phase } T < T_c \\ \xi \neq 0 \text{ at criticality } T = T_c \end{cases}$

NB : if TTI (equilibrium criticality), then $\xi = 0$.

Dynamical symmetry I : Schrödinger algebra $\mathfrak{sch}(d)$

dynamical symmetries of Langevin equation (deterministic part **!**)

Schrödinger operator in d space dimensions : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_{\mathbf{r}} \cdot \partial_{\mathbf{r}}$

(free) Schrödinger/heat equation
(noiseless) Edwards-Wilkinson equation } : $\mathcal{S}\phi = 0$

$$
\begin{aligned}
[\mathcal{S}, \mathbf{Y}_{\pm 1/2}] &= [\mathcal{S}, M_0] = [\mathcal{S}, X_{-1}] = [\mathcal{S}, \mathcal{R}] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S} + 2\mathcal{M}\left(x - \frac{d}{2}\right)
\end{aligned}
$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\mathcal{X} \in \mathfrak{sch}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$ and $x = x_\phi = \frac{d}{2}$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. Lie 1881, Niederer '72

$\mathfrak{sch}(d)$ maps solutions of $\mathcal{S}\phi = 0$ onto solutions.

Dynamical symmetry II : ageing algebra $\mathfrak{age}(d)$

$1D$ Schrödinger operator : $\mathcal{S} = 2\mathcal{M}\partial_t - \partial_r^2 + 2\mathcal{M}\left(x + \xi - \frac{1}{2}\right) t^{-1}$

generalised 'Schrödinger equation' : $\boxed{\mathcal{S}\phi = 0}$

extra potential term arises in several models, **without** time-translations

(e.g. $1D$ Glauber-Ising, spherical & Arcetri models)

if time-translations ($X_{-1} = -\partial_t$) are included, then $\xi = 0$

$$
\begin{aligned}
[\mathcal{S}, Y_{\pm 1/2}] &= [\mathcal{S}, M_0] = 0 \\
[\mathcal{S}, X_0] &= -\mathcal{S} \\
[\mathcal{S}, X_1] &= -2t\mathcal{S}
\end{aligned}
$$

infinitesimal change : $\delta\phi = \varepsilon\mathcal{X}\phi$, $\mathcal{X} \in \mathfrak{age}(d), |\varepsilon| \ll 1$

Lemma : If $\mathcal{S}\phi = 0$, then $\mathcal{S}(\mathcal{X}\phi) = 0$. NIEDERER '74 ; MH & STOIMENOV '11

$\boxed{\mathfrak{age}(d) \text{ maps solutions of } \mathcal{S}\phi = 0 \text{ onto solutions}}$.

Example for the t^{-1}-term in Langevin eq. : **Arcetri I model**

continuous slopes $u_{\mathbf{i}} \in \mathbb{R}^d$, replace RSOS condition by 'spherical' constraint
for $d > 0$ phase transition $T_c(d) > 0$, exponents not mean-field if $d < 2$

spherical constraint : $\left\langle \sum_{\mathbf{i}\in\Lambda} u_{\mathbf{i}}^2 \right\rangle = d\mathcal{N}$ MH & DURANG 15, MH 15

Langevin equation, with Lagrange multiplier $\mathfrak{z}(t)$ & centered gaussian noise $\eta_{\mathbf{i}}(t)$

$$\frac{\partial u_a(t,\mathbf{r})}{\partial t} = \nu \Delta u_a(t,\mathbf{r}) + \mathfrak{z}(t) u_a(t,\mathbf{r}) + \partial_a \eta(t,\mathbf{r}) \;\;,\;\; \langle \eta(t,\mathbf{r})\eta(s,\mathbf{r}')\rangle = 2\nu T \delta(t-s)\delta(\mathbf{r}-\mathbf{r}')$$

set $g(t) := \exp\left(2\int_0^t \mathrm{d}t'\, \mathfrak{z}(t')\right)$, spherical constraint gives Volterra eq.

$$g(t) = f(t) + 2T\int_0^t \mathrm{d}\tau\, f(t-\tau)g(\tau) \;\;,\;\; f(t) = \frac{d e^{-4t} I_1(4t)}{4t}\left(e^{-4t} I_0(4t)\right)^{d-1}$$

find for $T \le T_c$: $g(t) \stackrel{t\to\infty}{\sim} t^{-F} \Leftrightarrow \boxed{\mathfrak{z}(t) \sim \frac{F}{2} t^{-1}}$

quite analogous to spherical model of a ferromagnet

GODRÈCHE & LUCK 00
PICONE & MH 04

Schrödinger- & ageing-covariant two-point functions

two-point function $R = R(t,s;\mathbf{r}_1-\mathbf{r}_2) := \langle \phi_1(t,\mathbf{r}_1)\widetilde{\phi}_2(s,\mathbf{r}_2)\rangle$

Each ϕ_i characterized by (i) scaling dimensions x_i, ξ_i (ii) mass $\mathcal{M}_i$

* from Schrödinger-invariance

$$R(t,s,\mathbf{r}) = r_0 \delta_{x_1,x_2}\, s^{-1-a}\left(\frac{t}{s}-1\right)^{-1-a}\exp\left[-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right]$$

* from **ageing-invariance**

$$R(t,s;\mathbf{r}) = r_0 s^{-1-a}\left(\frac{t}{s}\right)^{1+a'-\lambda_R/2}\left(\frac{t}{s}-1\right)^{-1-a'}\exp\left(-\frac{\mathcal{M}_1}{2}\frac{\mathbf{r}^2}{t-s}\right)$$

with $1+a=\frac{x_1+x_2}{2}$, $a'-a=\xi_1+\xi_2$, $\lambda_R = 2(x_1+\xi_1)$, $\underbrace{\mathcal{M}_1+\mathcal{M}_2=0}_{\text{Bargman rule}}$

Example : $1D$ **Glauber-Ising model**, $T=0$, ϕ **: magnetisation**

$$x=\frac{1}{2},\ \widetilde{x}=0,\ \xi=0,\ \widetilde{\xi}=\frac{1}{4}.$$

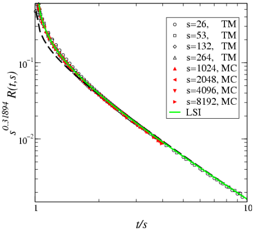
can derive **causality condition** $t>s$ from algebra extension by N

$\Rightarrow$ R is physically a **response function**.

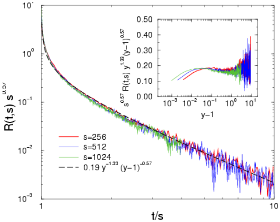
MH & UNTERBERGER 03, MH 14

Particle models : comparison of $R(t,s)$ with LSI-prediction :

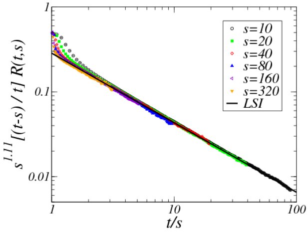
contact process (CP)



nonequil. kinetic Ising (PC)



voter Potts-3 (VP3)



CP : $a' - a \simeq 0.27$

PC : $a' - a \simeq 0.00(1)$

VP3 : $a' - a \simeq -0.1$

MH, Enss, Pleimling 06
Enss 06 ; Hinrichsen 06

Ódor 06

Chatelain, Tomé, de Oliveira 11

? is this good general agreement already conclusive ?

Observation : the **hidden assumption** $a = a'$, uncritically taken over from equilibrium, is often **invalid** out of equilibrium. Observables **cannot** always be identified with scaling operators.

$1D$ **KPZ** : find $R(t,s) = \left\langle \psi(t)\widetilde{\psi}(s)\right\rangle$ from 'logarithmic partner' of order parameter (ψ,ϕ) MH 13

scaling dimensions become **Jordan matrices** $\begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$, $\begin{pmatrix} \xi & \xi' \\ 0 & \xi \end{pmatrix}$ and similarly for response fields

* good collapse $\Rightarrow$ **no** logarithmic corrections $\Rightarrow$ $\boxed{x' = \widetilde{x}' = 0}$

* **no** logarithmic factors for $y \gg 1 \Rightarrow$ $\boxed{\xi' = 0}$

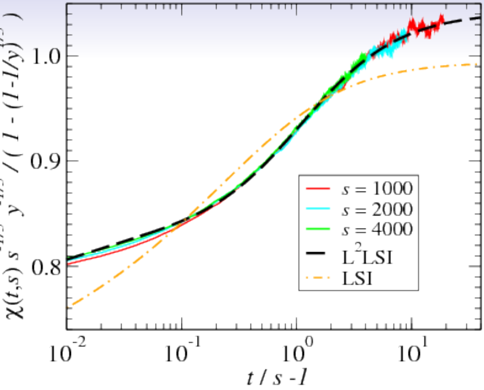
$\Rightarrow$ only $\widetilde{\xi}' = 1$ remains

$$f_R(y) = y^{-\lambda_R/z}\left(1-\frac{1}{y}\right)^{-1-a'}\left[h_0 - g_0 \ln\left(1-\frac{1}{y}\right) - \frac{1}{2} f_0 \ln^2\left(1-\frac{1}{y}\right)\right]$$

find integrated autoresponse $\chi(t,s) = \int_0^s \mathrm{d}u\, R(t,u) = s^{1/3} f_\chi(t/s)$

$$f_\chi(y) = y^{1/3}\left\{A_0\left[1-\left(1-\frac{1}{y}\right)^{-a'}\right] + \left(1-\frac{1}{y}\right)^{-a'}\left[A_1 \ln\left(1-\frac{1}{y}\right) + A_2 \ln^2\left(1-\frac{1}{y}\right)\right]\right\}$$

with free parameters A_0, A_1, A_2 and a' — for the $1D$ KPZ class, use $\frac{\lambda_R}{z} - a = 1$



MH, NOH, PLEIMLING 12

non-log LSI with $a = a'$:
deviations $\approx 20\%$

non-log LSI with $a \neq a'$:
works up to $\approx 5\%$

log LSI : works **better than** $\approx 0.1\%$

R	a'	A_0	A_1	A_2
$\langle\phi\widetilde{\phi}\rangle$ – LSI	-0.500	0.662	0	0
$\langle\phi\widetilde{\psi}\rangle$ – L^{1}LSI	-0.500	0.663	$-6 \cdot 10^{-4}$	0
$\langle\psi\widetilde{\psi}\rangle$ – L^{2}LSI	-0.8206	0.7187	0.2424	-0.09087

logarithmic LSI fits data at least down to $y \simeq 1.01$, with
$a' - a \approx -0.4873$ (can we make a conjecture ?)

#extensions to $z \in \mathbb{R}$:
requires use of fractional derivatives, qualitatively similar
infinitesimal generators become non-local integral operators
applications to models with conserved order-parameter ('model B')

#a distinct universality class :
Diffusion-limited erosion KRUG & MEAKIN 86

$$\partial_t h = \nu \left(\nabla \cdot \nabla\right)^{1/2} h + \eta$$

non-local equation of motion, dynamical exponent $z = 1$
dynamical symmetry given by **non-local representation** of an
unconventional conformal algebra
reproduces exactly known two-time response function

5. Conclusions

* long-time dynamics of growing interfaces naturally evolves towards dynamical scaling & ageing
* phenomenology very similar to ageing phenomena in simple magnets, glasses, frustrated spin systems, particle-reaction models and polymer collapse   MAJUMDER & JANKE 16
* subtleties in the precise scaling forms & space-dependent profiles
* what are the necessary ingredients to derive the shape of two-time response functions, . . . ?
* derivation of causality through extension of Cartan sub-algebra

* several unsolved physical and mathematical questions

proving extended dynamical symmetries can remain delicate !

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