Old, new and recent results concerning the perturbed Potts model

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Table of contents

The 3D random Ising model

Classical Ising model on a cubic lattice

$$H = -J\sum_{(i,j)} \sigma_i\sigma_j - h\sum_i \sigma_i \qquad \sigma_i = \pm 1$$

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where $\epsilon_{ij} \in \{0; 1\}$ is a random variable.

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where $\epsilon_{ij} \in \{0; 1\}$ is a random variable.

$$\frac{\beta}{\nu} \simeq 0.515(5), \qquad \frac{\gamma}{\nu} \simeq 1.97(2), \qquad \nu \simeq 0.68(2).$$

New universality class (as predicted by Harris criterion).

The 3D random 4-state Potts model

Classical "spins" lying on the nodes of a cubic lattice:

$$H = -J \sum_{(i,j)} \delta_{\sigma_i,\sigma_j} - h \sum_i \delta_{\sigma_i,0} \qquad \sigma_i = 0, \dots, q-1$$

The 3D random 4-state Potts model

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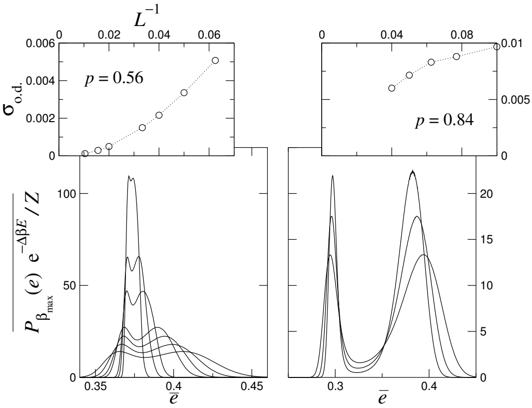

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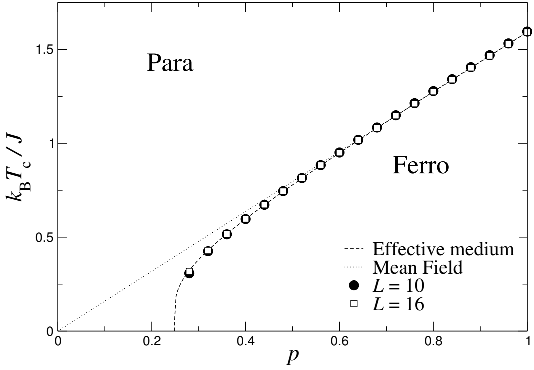


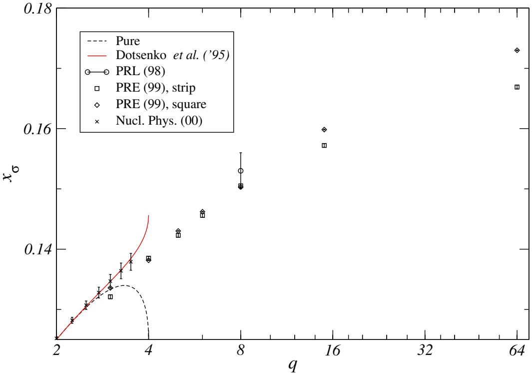
The 3D random q-state Potts model

A q-dependent random fixed point

$$q = 2 \qquad \frac{\beta}{\nu} = 0.515(5)$$
$$q = 3 \qquad \frac{\beta}{\nu} = 0.539(2)$$
$$q = 4 \qquad \frac{\beta}{\nu} = 0.73(2)$$

The 3D random q-state Potts model

A q-dependent random fixed point **(as in 2D)**



Correlated disorder

Algebraically correlated couplings

$$\overline{(J_{ij}-\bar{J})(J_{kl}-\bar{J})} \sim ||\vec{r}_{ij}-\vec{r}_{kl}||^{-a}$$

Disorder is relevant when $a < d$ and

$$\nu < \frac{2}{a}$$

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$$\nu < \frac{2}{a}$$

RG study of the n-components ϕ^4 model in dimension $d = 4 - \varepsilon$ leads to a **new fixed point** (Weinrib-Halperin):

$$\nu = \frac{2}{a} \quad \text{(exact)}, \qquad \eta = \mathcal{O}(\varepsilon^2).$$

Recent results for the 2D Ising model (a close to 2)

(M. Dudka, A.A. Fedorenko, V. Blavatska, Y. Holovatch, arXiv:1602.07229)

$$\nu = \frac{2}{a} + \mathcal{O}((2-a)^3), \qquad \frac{1}{4} - \frac{2-a}{8} \leq \eta \leq \frac{1}{4}$$

Compatible with Monte Carlo simulations for the 3D Ising model.

Numerical generation of configurations of correlated random couplings

Step one: Simulate the 2D Ashkin-Teller model

$$-\beta H^{\rm AT} = \sum_{(i,j)} \left[J^{\rm AT}\sigma_i\sigma_j + J^{\rm AT}\tau_i\tau_j + K^{\rm AT}\sigma_i\sigma_j\tau_i\tau_j \right]$$

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Two broken $\mathbb{Z}_2$-symetries in the low-temperature phase. Order parameters: magnetization $\sum_i \sigma_i$ and polarization $\sum_i \sigma_i\tau_i$.

Self-dual critical line with varying critical exponents:

$$\beta_\sigma^{\rm AT}=\frac{2-y}{24-16y},\quad \beta_{\sigma\tau}^{\rm AT}=\frac{1}{12-8y},\quad \nu^{\rm AT}=\frac{2-y}{3-2y}$$

where $y\in[0;4/3]$ and $\cos\frac{\pi y}{2}=\frac{1}{2}\left[e^{4K^{\rm AT}}-1\right]$

Polarisation-polarisation correlation function of the AT model:

$$\langle \sigma_i \tau_i \, \sigma_j \tau_j \rangle \sim |\vec{r}_i - \vec{r}_j|^{-2\beta_{\sigma\tau}^{\rm AT}/\nu^{\rm AT}}$$

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Step two: Generate Ashkin-Teller spin configurations and associate a coupling configuration of the Potts model to each of them by

$$J_{ij} = \frac{J_1 + J_2}{2} + \frac{J_1 - J_2}{2} \sigma_i \tau_i \in \{J_1, J_2\},$$

so that

$$\overline{(J_{ij} - \bar{J})(J_{kl} - \bar{J})} \sim |\vec{r}_i - \vec{r}_k|^{-a}$$

with $a = 2\beta_{\sigma\tau}^{\rm AT}/\nu^{\rm AT}$. Self-duality of the random Potts model is preserved.

$$-\beta H = \sum_{(i,j)} J_{ij} \delta_{s_i, s_j}$$

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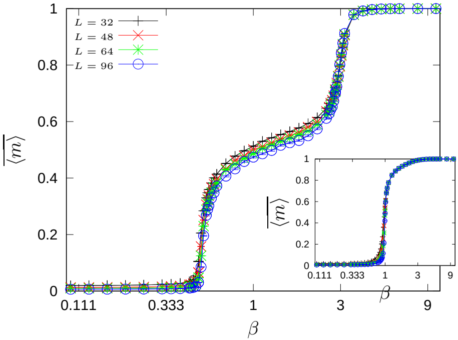
BUT a is small and far from $a=2$.

Temperature behaviour of the 8-state Potts model



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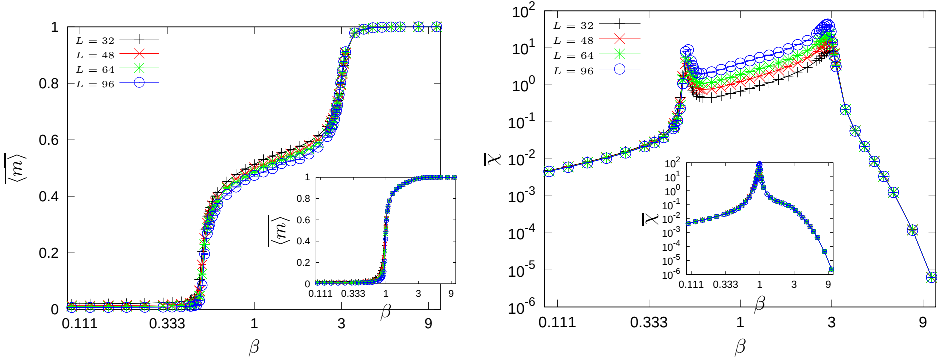

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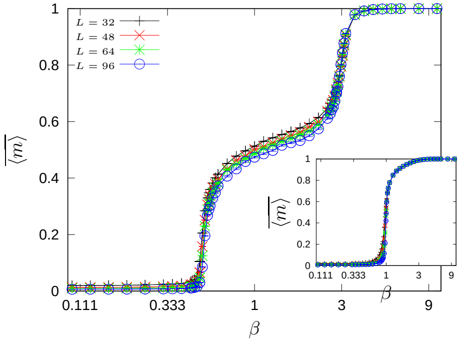
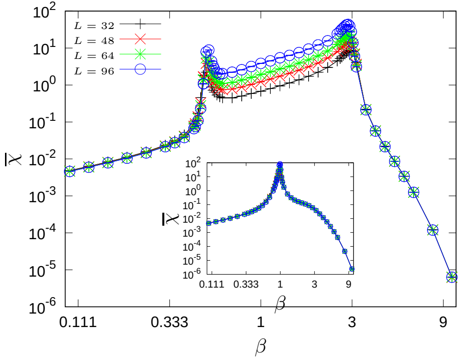
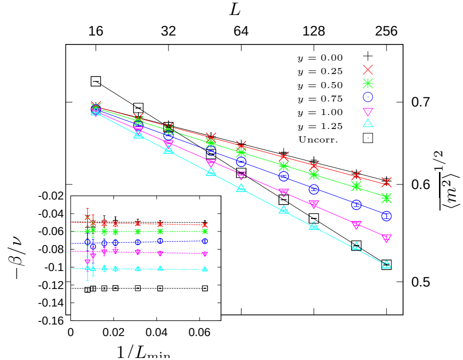




Griffiths phase!

Singularity of free energy in a finite range of temperatures, due to the existence of macroscopic regions with a high concentration of strong couplings and acting as super-paramagnets.

Algebraic Finite-Size Scaling in the Griffiths region:



Critical exponent β/ν depends on disorder correlations (exponent a) and is compatible with the bound $\eta \leq 1/4$. Stable with disorder strength $r = J_1/J_2$.

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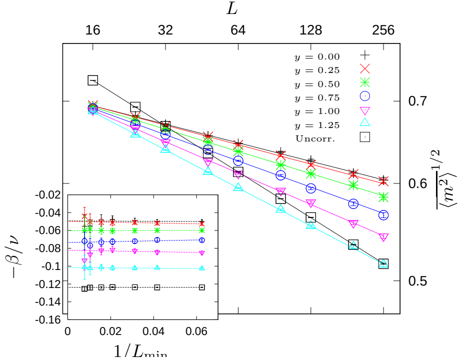

Critical exponent β/ν depends on disorder correlations (exponent a) and is compatible with the bound $\eta \leq 1/4$. Stable with disorder strength $r = J_1/J_2$.

Independent of the number q of Potts states! ($q = 2$ to 16 tested)

Self-averaging ratio (sample-to-sample relative fluctuations)

$$R_m = \frac{\overline{\langle m\rangle^2} - \overline{\langle m\rangle}^2}{\overline{\langle m\rangle}^2}$$

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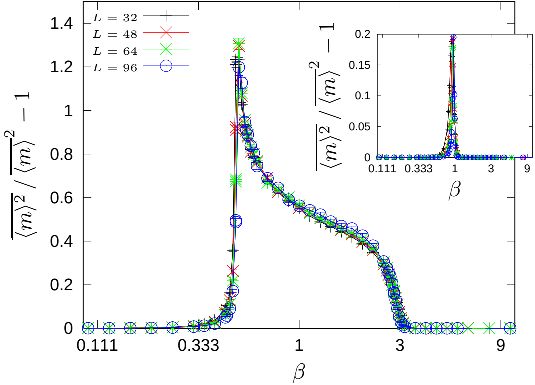

Magnetization is non-self averaging in the Griffiths region.

$$R_m = \mathrm{Cst} \implies \chi^* = L^d \left[\overline{\langle m\rangle^2} - \overline{\langle m\rangle}^2\right] = L^d R_m \overline{\langle m\rangle}^2 \sim L^{d-2\beta/\nu}$$

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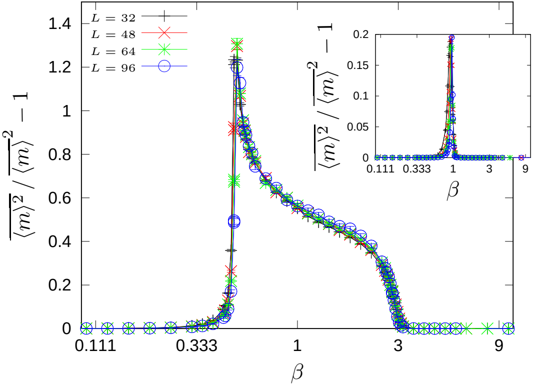

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Hyperscaling is violated for $\bar{\chi} = \overline{\langle m\rangle^2} - \overline{\langle m\rangle^2}$ but satisfied for χ^*!

In the same way, $\nu \gg 1$ but $\nu^* \simeq 2/a$.

Another random fixed-point has similar properties:

- independent of the number of states q
- hyperscaling violation

$$\gamma/\nu + 2\beta/\nu \neq d$$

- Griffiths phase.

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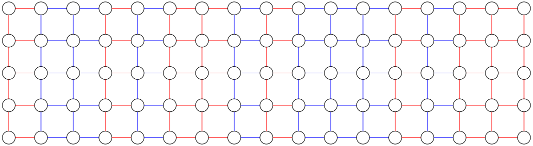
... the **Potts McCoy-Wu model**.

$$-\beta H = \sum_{x,y} J_x \left(\delta_{\sigma_{x,y},\sigma_{x+1,y}} + \delta_{\sigma_{x,y},\sigma_{x,y+1}}\right)$$



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Critical behavior determined by disorder fluctuations (thermal fluctuations irrelevant). Inacessible by perturbative RG but asymptotically exact results using Strong Disorder Renormalization Group (Fisher). Number of Potts states q shown to be irrelevant (Senthil, Majumdar).

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Partial summary

- Homogeneous disorder leads a new q-dependent random fixed point
- Perturbative RG finds another random fixed point with algebraically correlated disorder ($\nu = 2/a$)
- A different (q-independent) infinite-disorder fixed point in the McCoy-Wu model (infinitely correlated in one dimension)
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Old problem/New question:
What about aperiodic modulation of the couplings ?

- Monte Carlo simulations observed q-dependent critical exponents
- SDRG results were recently shown to give results for the Ising model comptible with free fermions techniques.

More convenient to study the anisotropic extreme limit:

$$-\beta H = \sum_{x,y} \big(J_h(x)\delta_{\sigma_{x,y},\sigma_{x+1,y}} + J_v(x)\delta_{\sigma_{x,y},\sigma_{x,y+1}} \big), \quad J_h \to +\infty, J_v \to 0$$

The transfer matrix tends towards a quantum evolution operator

$$\begin{aligned} T(\sigma_1',\sigma_2',\ldots;\sigma_1,\sigma_2,\ldots) \quad &= \quad e^{\beta\sum_i \left[\frac{J_v}{2}\delta_{\sigma_i',\sigma_{i+1}'} + \frac{J_v}{2}\delta_{\sigma_i,\sigma_{i+1}} + J_h\delta_{\sigma_i,\sigma_i'}\right]} \\ &\longrightarrow \quad \langle \sigma_1',\sigma_2',\ldots|e^{-\beta H}|\sigma_1,\sigma_2,\ldots\rangle \end{aligned}$$

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with the quantum Potts hamiltonian

$$\hat{H} = -\sum_i \sum_{\sigma=1}^{q-1} \big[J_i (\hat{\Omega}_i)^\sigma (\hat{\Omega}_{i+1})^{-\sigma} + h_i \hat{N}_i^\sigma \big]$$

where, for $q = 4$ for instance,

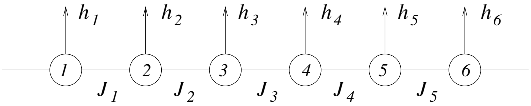
$$\hat{N} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \qquad \hat{\Omega}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & \omega^3 \end{pmatrix}$$

and

$$\omega = e^{\frac{2i\pi}{q}}$$

Strong Disorder Renormalization Group

$$\hat{H} = -\sum_i \sum_{\sigma=1}^{q-1} \left[J_i (\hat{\Omega}_i)^\sigma (\hat{\Omega}_{i+1})^{-\sigma} + h_i \hat{N}_i^\sigma \right]$$



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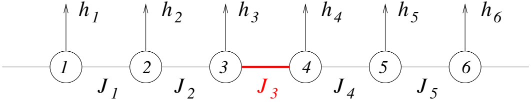

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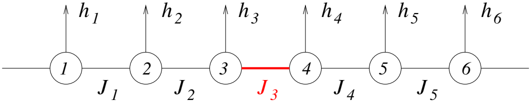

Step two: Find the q ground states of $-J_3 \sum_\sigma (\hat{\Omega}_3)^\sigma (\hat{\Omega}_4)^{-\sigma}$:

$$\{|00\rangle, |11\rangle, \ldots, |q-1, q-1\rangle\}$$

Replace the two spins σ_3 and σ_4 by an effective Potts macro-spin.

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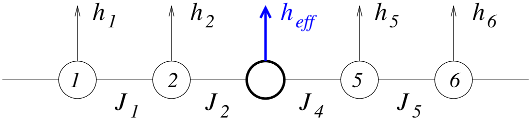
Replace the two spins σ_3 and σ_4 by an effective Potts macro-spin.
Step three: Use perturbation theory to compute an effective transverse field acting on the new macro-spin

$$h^{\mathrm{eff}} = \frac{2h_3 h_4}{qJ_3}$$



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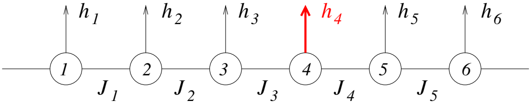
Similarly, a strong transverse field freezes a spin.



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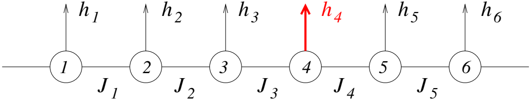
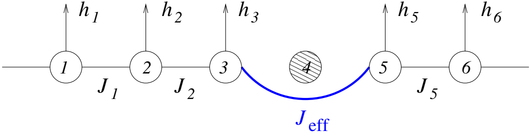
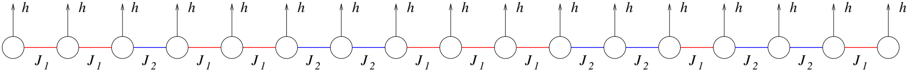


and an effective coupling between the nearest spins is induced

$$J^{\text{eff}} = \frac{2 J_3 J_4}{q h_4}$$



The Paper-Folding sequence:



Marginal for $q = 2$ (Ising) and relevant perturbation for $q > 2$ (Luck criterion).

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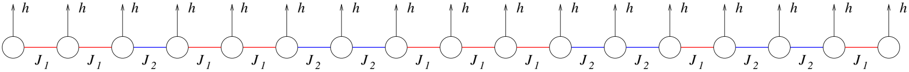

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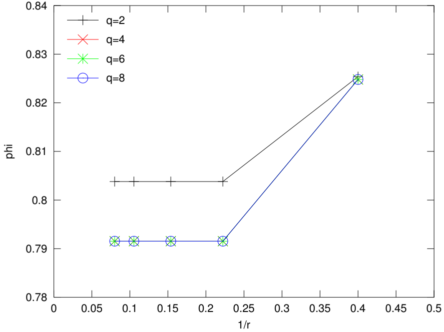
Magnetic scaling dimension $\phi = 1 - \beta/\nu$ independent of q for $q > 2$ and $r = J_1/J_2 > 2.5$ (contradicts Monte Carlo simulations!)



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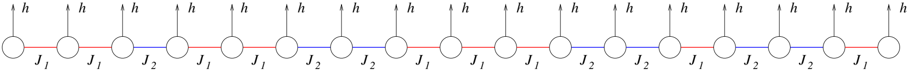

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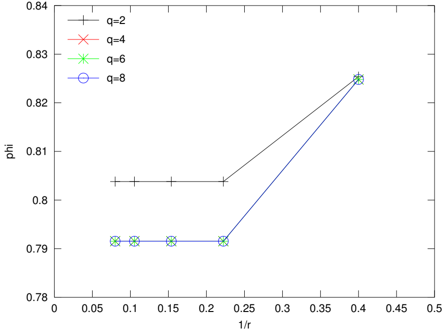
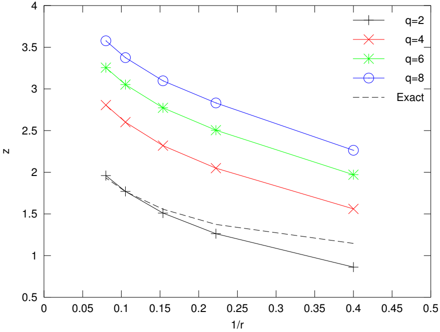
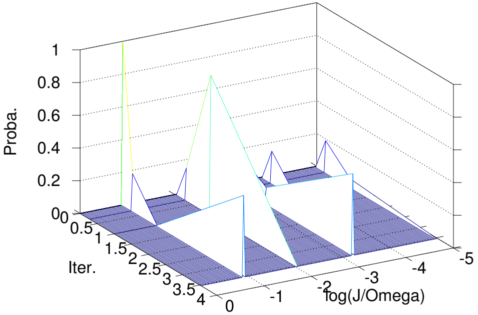




BUT dynamical exponent z depends on q !

Can we trust SDRG?

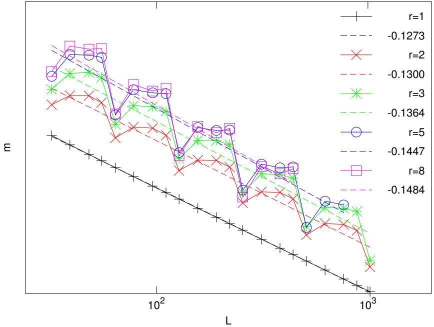
SDRG works only if couplings are broadly distributed at the fixed point (for perturbation theory to hold).



The ratio between the largest and second-largest couplings does not increase! Not an infinite-disorder fixed point!?

Density Matrix Renormalization Group

(Purely numerical, thermal fluctuations properly taken into account)
Large log-periodic oscillations.



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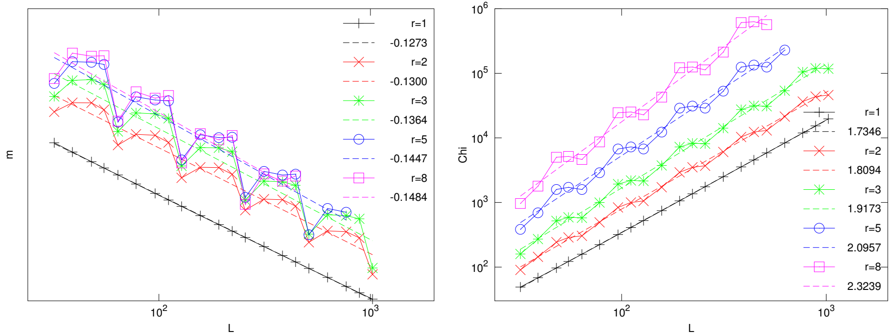
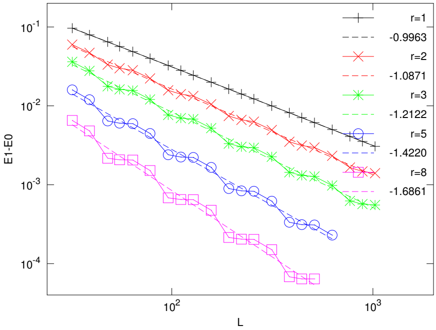
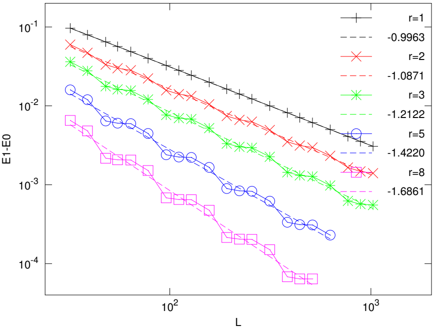
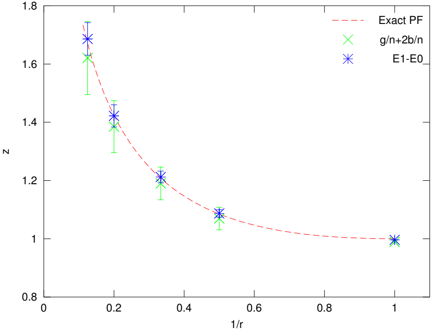


Magnetic scaling dimension given by SDRG recovered only in the limit $r = J_1/J_2 \longrightarrow +\infty$.


r=1
-0.9963
r=2
-1.0871
r=3
-1.2122
r=5
-1.4220
r=8
-1.6861
10^{-1}
10^{-2}
10^{-3}
10^{-4}
E1-E0
10^2
10^3
L





$$\gamma/\nu + 2\beta/\nu = 1 + z$$

DMRG is in agreement with free fermions calculations for any r.

Preliminary conclusions

For the Ising model

- Paperfolding seq. is marginal: not an infinite-disorder fixed point but a line of fixed points,
- DMRG agrees with free fermions calculations,
- SDRG predicts correct dynamical exponent for $r \gg 1$ and scaling dimension only when $r \longrightarrow +\infty$

For the $q > 2$ Potts model

- Similar behavior of SDRG flow than with Ising model. No flow towards an infinite-disorder fixed point ?!
- Dynamical exponent z depends on r and q.
- But q-independent magnetic exponents.
- DMRG calculations will eventually converge...